Projection Residual Geometry

Global Structure of Admissibility in Closure-First Relational Frameworks

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Abstract

Projection Residual Geometry (PRG) formulates admissibility in closure-first relational frameworks as projection onto a constraint-defined manifold in configuration space. In this work we clarify the global geometric structure implied by PRG.

We show that the projection residual defines a scalar field over configuration space measuring deviation from closure-compatible structure, and that admissible configurations correspond to the minimum of this field. The capacity tensor determines the geometry of deviation, yielding a warped Grassmannian metric and a natural notion of relational distance.

We emphasize that PRG provides a global geometric characterization of admissibility and deviation, without introducing additional dynamical or differential structure. More refined geometric constructions—such as connection, curvature, and holonomy—arise only at the level of local closure restoration and are developed separately in *Closure-Restoration Geometry (CRG)* [1].

Thus, PRG establishes the global projection structure of admissibility, while CRG develops its local geometric refinement.

Contents

1. Introduction

Projection Residual Geometry (PRG) provides a geometric formulation of admissibility in closure-first relational frameworks. In this setting, physical configurations are not generated dynamically, but are defined by global consistency conditions applied all-at-once to the entire configuration. Admissibility is therefore not a matter of evolution, but of satisfaction of constraint. This work assumes the closure-first axioms defined in *Axioms of Ontic Minimalism (AOM)* [2].

The central question addressed by PRG is:

Given an arbitrary configuration, how is its admissibility determined?

The answer is minimal and structural. Let $\mathcal{M}$ denote the set of configurations satisfying the governing closure constraints. Then admissibility is characterized by projection onto $\mathcal{M}$, and deviation from admissibility is quantified by the minimal distance to $\mathcal{M}$ under an appropriate metric.

Thus PRG introduces three core objects:

- the **admissible manifold** $\mathcal{M}$, defined by global closure constraints,
- the **projection operator** $P_{\mathcal{M}}$, mapping arbitrary configurations to admissible ones,
- the **projection residual** $R(C)$, measuring minimal deviation from admissibility.

This construction does not introduce additional physical structure. It expresses, in geometric form, the requirement that admissibility depends only on constraint satisfaction and not on any underlying dynamical process.

1.1 From Closure to Projection

In closure-first frameworks, admissibility is defined by a constraint map

$$\Gamma(C) = 0, \qquad (1.1)$$

where C is a configuration in an ambient configuration space $\mathcal{C}$. The admissible manifold is therefore

$$\mathcal{M} := \{C \in \mathcal{C} \mid \Gamma(C) = 0\}. \qquad (1.2)$$

An arbitrary configuration $C \notin \mathcal{M}$ is not “evolved” into admissibility. Instead, it is evaluated relative to $\mathcal{M}$. The admissible representative is the element of $\mathcal{M}$ that minimizes distance from C under the chosen metric, and the residual is the minimal deviation required to satisfy closure.

This projection-based characterization is the minimal structure required to relate arbitrary configurations to admissibility. Any alternative formulation would require additional operators or dynamical rules not implied by global closure.

1.2 Configuration Space and Metric Structure

Relational configurations are represented as subspaces of a higher-dimensional relational state space. This naturally leads to a Grassmannian configuration space

$$V \in \mathrm{Gr}_{\kappa}(k, n), \tag{1.3}$$

where each point corresponds to a k-dimensional relational subspace in an n-dimensional ambient space. The Grassmannian $Gr(k, n)$is the manifold of k-dimensional subspaces of $\mathbb{R}^n$. [3]

Let the capacity tensor κ define a weighted inner product on the ambient space:

$$\langle x, y \rangle_{\kappa} = x^T \kappa y \tag{1.4}$$

The κ-weighted inner product induces a corresponding metric on the ambient space, and orthonormal bases used to compute principal angles are defined with respect to this metric. Principal angles between subspaces V and W are then computed with respect to this κ-weighted inner product, yielding angles $\widetilde{\theta_1}$.

The Grassmannian distance is defined as:

$$d_{\kappa}^2(V, W) = \sum_{i=1}^{k} \widetilde{\theta_i^2} \tag{1.5}$$

Principal angles provide the canonical notion of distance between subspaces. [4] In this formulation, the capacity tensor κ enters through the geometry of the inner product itself, rather than as a post hoc weighting of principal angles. This ensures that the distance is invariant under changes of basis that preserve the κ-weighted inner product, and is therefore geometrically well-defined.

The admissible configurations form a subset

$$\mathcal{M} \subset \mathrm{Gr}_{\kappa}(k, n), \tag{1.6}$$

defined by closure constraints.

This metric structure is not fixed at the level of principle; it is representation-dependent. What is required is only that configuration space admit a κ-compatible distance structure sufficient to define projection onto the admissible manifold.

1.3 Projection Residual as Geometric Structure

The projection residual is defined as

$$R(C) = \inf_{X \in \mathcal{M}} d(C, X), \tag{1.7}$$

or equivalently,

$$R(C) = d(C, P_{\mathcal{M}}(C)), \tag{1.8}$$

when a minimizing projection exists.

The residual is therefore not an auxiliary quantity. It is the minimal geometric measure of deviation from admissibility: configurations in $\mathcal{M}$ have zero residual, while configurations outside $\mathcal{M}$ are characterized by their minimal deviation from closure compatibility.

This induces a scalar field over configuration space:

$$R: \mathcal{C} \to \mathbb{R}_{\geq 0}, \tag{1.9}$$

which we refer to as the **residual field**.

The residual field encodes the global structure of admissibility. Its minima correspond to admissible configurations, and its variation reflects how strongly configurations violate closure constraints. This corresponds to the standard notion of distance to a constraint manifold in geometric analysis. [5]

1.4 Minimality and Necessity

The projection structure introduced here is not arbitrary. It is the minimal formalization of admissibility consistent with closure-first principles:

- admissibility is defined globally (AGC),
- admissibility is independent of decomposition (NPRF),
- closure must be satisfied (BBP),
- deviation must be measured relative to closure satisfaction.

Projection onto $\mathcal{M}$ is the simplest structure satisfying these conditions. It introduces no additional ontology, no dynamical law, and no privileged representation.

In this sense, within representations that admit a metric structure compatible with closure constraints, PRG is not an optional modeling choice. It is the minimal projection-based formalization of constraint-defined admissibility within representations that admit an admissibility-relevant distance structure.

1.5 Scope and Relation to Further Geometry

The present work is intentionally limited to the **global geometric structure** of admissibility.

PRG defines:

- the admissible manifold $\mathcal{M}$,
- the projection operator $P_{\mathcal{M}}$,
- the residual field $R(C)$.

It does not, by itself, define:

- local transport within $\mathcal{M}$,
- connection structure,
- curvature or holonomy.

These arise when one examines how projection behaves locally near admissible configurations. That refinement is developed in CRG [1], where projection becomes a local restoration map and induces connection and curvature structure.

Thus:

$$\begin{aligned} &\text{PRG} \;\to\; \text{global admissibility structure} \\ &\text{CRG} \;\to\; \text{local geometric refinement} \end{aligned}$$

The two are consistent: CRG provides the first-order local realization of the global projection structure defined here. PRG therefore defines only the global geometry of admissibility; any differential or transport structure arises only from local analysis of projection behavior, as developed in CRG.

1.6 Interpretation

PRG replaces dynamical notions of correction or evolution with geometric deviation from constraint satisfaction.

Configurations do not evolve toward admissibility. They are evaluated by their position relative to the admissible manifold. The residual quantifies how far a configuration lies from satisfying the governing constraints.

In this framework:

- admissibility is geometric,
- deviation is metric,
- and physical structure is defined entirely by constraint satisfaction.

This provides a global, representation-neutral foundation for closure-first relational theories.

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2. Formal Projection Structure and Residual Geometry

Having introduced the projection-based characterization of admissibility, we now formalize the structure of PRG.

2.1 Admissible Manifold

Let the ambient configuration space be a finite-dimensional vector space

$$\mathcal{C} \cong \mathbb{R}^N, \tag{2.1}$$

and let admissibility be defined by a closure map

$$\Gamma: \mathcal{C} \to \mathbb{R}^m, \tag{2.2}$$

such that admissible configurations satisfy

$$\Gamma(C) = 0. \tag{2.3}$$

The admissible manifold is therefore

$$\mathcal{M} := \{C \in \mathcal{C} \mid \Gamma(C) = 0\}. \tag{2.4}$$

This formulation expresses the Adynamical Global Constraint (AGC) principle: admissibility is determined by global constraint satisfaction, not by dynamical evolution.

2.2 Projection onto the Admissible Set

Given an arbitrary configuration $C \in \mathcal{C}$, the admissible representative is defined by projection onto $\mathcal{M}$:

$$P_{\mathcal{M}}(C) := \arg\min_{X \in \mathcal{M}} d(C, X), \tag{2.5}$$

where $d(C, X)$ is a distance function on configuration space. [6]

This projection exists and is unique when $\mathcal{M}$ is nonempty, closed, and the metric space is complete; otherwise the residual is defined via the infimum. The projection is defined relative to a chosen representation of configuration space. The existence and uniqueness of $P_{\mathcal{M}}(C)$ depend on:

- non-emptiness of $\mathcal{M}$,

- regularity of the constraint set,
- and properties of the distance function.

When the minimizer is not unique, $P_{\mathcal{M}}(C)$ denotes any minimizing element, and the residual is defined via the infimum distance. Non-emptiness of $\mathcal{M}$ is a condition on the closure map Γ; it is assumed throughout and is satisfied for all physically relevant configurations in the cRBW corpus.

2.3 Residual Definition

The projection residual is defined as:

$$R(C) := \inf_{X \in \mathcal{M}} d(C, X). \tag{2.6}$$

When the projection exists:

$$R(C) = d(C, P_{\mathcal{M}}(C)). \tag{2.7}$$

The residual quantifies the minimal correction required to satisfy admissibility. It is therefore not a derived quantity but the **primary measure of deviation from closure**.

2.4 Residual Field

The residual defines a scalar field over configuration space:

$$R: \mathcal{C} \to \mathbb{R}_{\geq 0}. \tag{2.8}$$

Away from singular points where the minimizing projection is nonunique, the residual field is continuous and differentiable in a neighborhood of $\mathcal{M}$. This field has the following properties:

- $R(C) = 0$ if and only if $C \in \mathcal{M}$,
- $R(C) > 0$ otherwise,
- $R(C)$ measures minimal deviation from admissibility.

Thus, admissibility can be equivalently characterized as the zero-level set of the residual field.

The residual field therefore encodes the global structure of admissibility independently of any dynamical interpretation.

2.5 Metric Structure

The distance function $d(C, X)$ encodes the geometry of deviation. In Grassmannian representations, this is given by a capacity-weighted principal-angle metric:

$$d_\kappa^2(\mathrm{V}, \mathrm{W}) = \sum_i \rho_i \theta_i^2, \tag{2.9}$$

where:

- θ_i are principal angles between subspaces,
- $\rho_i > 0$ are the eigenvalues of the capacity tensor κ in the principal basis.

More generally, the metric must satisfy:

- positivity,
- symmetry,
- and compatibility with the representation of configuration space.

The specific form of d is not fixed at the level of principle; it is determined within representations.

2.6 Capacity Weighting

The capacity tensor κ (or equivalently the weights ρ_i) determines the geometry of correction.

It specifies:

- which directions in configuration space are costly to adjust,
- which are weakly constrained,
- and how deviations propagate under projection.

In Grassmannian representations, κ is diagonal with entries ρ_i, recovering the warped principal-angle metric.

Thus, the metric structure is not independent of the physical content; it encodes the admissibility-relevant weighting of relational directions.

2.7 Projection Necessity

The projection formulation is not an arbitrary modeling choice.

Given:

- global admissibility (AGC),
- decomposition invariance (NPRF),
- closure (BBP),

any admissible mapping from arbitrary configurations must:

1. depend only on deviation from constraint satisfaction,

2. introduce no additional structure beyond admissibility,

3. be independent of decomposition.

Projection is the minimal structure satisfying these conditions within representations that admit a compatible metric structure.

PRG is therefore not merely convenient—it is the minimal projection-based formalization of constraint-defined admissibility within representations that admit an admissibility-relevant distance structure.

2.8 Representation Independence

The projection structure is independent of the specific mathematical representation.

Different frameworks may encode:

- configuration space,
- admissible set,
- and distance function

in different ways.

Examples include:

- simplicial complexes,
- algebraic relational encodings,
- Grassmannian configuration spaces.

Under Representational Neutrality (RN), these are encodings of the same constraint structure, not distinct ontologies.

2.9 Global–Local Relationship

The projection operator $P_{\mathcal{M}}$ defines admissibility at the global level.

In local neighborhoods near $\mathcal{M}$, projection admits a first-order approximation as a linear correction onto the tangent space of admissibility.

This local structure is developed in CRG [1], where projection becomes a local restoration map and induces connection and curvature structure.

Thus:

- PRG defines the global projection structure,
- CRG provides its local geometric realization.

2.10 Summary

Projection Residual Geometry provides a minimal geometric formulation of admissibility:

- admissible configurations lie on a constraint-defined manifold $\mathcal{M}$,
- arbitrary configurations are evaluated via projection onto $\mathcal{M}$,
- deviation is quantified by the residual $R(C)$,
- the residual defines a scalar field encoding admissibility structure.

No dynamical laws, correction processes, or underlying carriers are required.

PRG therefore establishes the global geometry of admissibility, forming the foundation upon which local geometric refinement (CRG) and physical consequences (e.g. quantum structure) are built.

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3. Residual Geometry on the Grassmannian

This section specifies the **global configuration geometry** used in PRG and shows how the projection residual appears as a **scalar field over relational configuration space**.

3.1 Configuration Space as a Grassmannian

In relational formulations, a configuration can be represented as a k-dimensional subspace of an n-dimensional relational state space. This identifies configuration space with a (capacity-weighted) Grassmannian:

$$V \in \mathrm{Gr}_\rho(k, n). \tag{3.1}$$

Each point V encodes a relational selection (e.g., a set of admissible relations, correlations, or subspace constraints) in the ambient space.

Under Representational Neutrality (RN), the Grassmannian is not ontological; it is a convenient encoding of relational structure that supports:

- comparison of configurations,
- projection onto admissible sets,
- and measurement of deviation.

In what follows, the residual may be expressed either as $R(C)$ in the ambient configuration space or as $R(V)$ in the Grassmannian representation. These are equivalent under the chosen encoding of configurations as relational subspaces.

3.2 Distance and Capacity Weighting

Distances between configurations $V, W \in \mathrm{Gr}_\rho(k,n)$ are measured using a capacity-weighted principal-angle metric:

$$d_\rho^2(V,W) = \sum_i \rho_i\ \theta_i^2, \tag{3.2}$$

where:

- θ_i are the principal angles between V and W,
- $\rho_i > 0$ are capacity weights.

The weights ρ_i encode the relative admissibility cost of deviations along different relational directions. In diagonal form, the capacity tensor κcorresponds to the matrix with entries ρ_i.

This metric defines the geometry in which projection and residual are evaluated.

3.3 Admissible Subset in Configuration Space

The closure constraints define an admissible subset:

$$\mathcal{M} \subset \mathrm{Gr}_\rho(k,n), \tag{3.3}$$

consisting of all configurations satisfying global closure.

The projection operator becomes:

$$P_{\mathcal{M}}(V) = \underset{W \in \mathcal{M}}{\arg\min}\ d_\rho(V,W). \tag{3.4}$$

The projection residual is then:

$$R(V) = d_\rho(V, P_{\mathcal{M}}(V)). \tag{3.5}$$

Thus admissibility is characterized entirely within configuration space geometry.

3.4 Residual as a Scalar Field

The projection residual defines a scalar field over the Grassmannian:

$$R: \mathrm{Gr}_\rho(k,n) \to \mathbb{R}_{\geq 0}. \tag{3.6}$$

This field has a direct geometric interpretation:

- $R(V) = 0$ → admissible configurations
- $R(V) > 0$ → closure-incompatible configurations

The admissible manifold $\mathcal{M}$ is therefore the zero-level set of the residual field.

The residual field encodes how strongly each configuration violates closure constraints and therefore provides a global geometric measure of admissibility.

3.5 Relation to Closure Geometry

In discrete relational realizations (e.g., simplicial structures), closure violation is often expressed through curvature-like quantities such as deficit angles ϵ_h. In such settings, the projection residual is proportional to these quantities:

$$R(V) \approx \epsilon_h \tag{3.7}$$

Equ 3.7 is for the near-admissible regime, up to representation-dependent scaling. The proportionality depends on the chosen metric and normalization.

This relation should be understood as representational:

- deficit angles encode closure failure in a simplicial representation,
- the residual encodes closure failure in configuration space.

The two are consistent descriptions of the same underlying constraint violation.

3.6 Compatibility Structure

The residual field induces a natural notion of compatibility between configurations.

A representative compatibility kernel may be defined as:

$$K\left(V_i, V_j\right) \sim \exp!\left(-\eta\, d_\kappa^2\left(V_i, V_j\right)\right). \tag{3.8}$$

This kernel measures relational compatibility:

- small residual difference → strong compatibility
- large residual difference → weak compatibility

This structure arises directly from projection geometry and does not require dynamical interpretation. It encodes how relational configurations align relative to admissibility.

This kernel structure is a provisional construction motivated by the residual geometry; its full derivation from AGC/LPS constraints and its role in the broader corpus are reserved for subsequent development. Under RN, it should be understood as one possible encoding, not a required structure. Such kernel constructions are standard in geometric and manifold-learning contexts, where distance-based kernels induce spectral operators on configuration space. [7]

3.7 Global Geometric Interpretation

The Grassmannian formulation allows PRG to be viewed as a **global geometry of admissibility**:

Object	Role
$\mathrm{Gr}_\rho(k, n)$	configuration space
$\mathcal{M}$	admissible subset
$R(V)$	deviation from admissibility
$P_{\mathcal{M}}$	projection to admissibility
$K(V_i, V_j)$	compatibility structure

This structure is purely geometric:

- no dynamical evolution is introduced,
- no local transport is assumed,
- no curvature or connection is required.

All structure arises from projection relative to closure constraints.

3.8 Scope Limitation

The structures described here are global. They do not specify:

- how projection behaves locally,
- how corrections vary across configuration space,
- or how path-dependent effects arise.

Such questions require analysis of local projection behavior near $\mathcal{M}$. This leads to a refinement in which projection becomes a local restoration map and induces connection and curvature structure.

That refinement is developed in Closure-Restoration Geometry (CRG).

3.9 Summary

The Grassmannian formulation makes explicit that PRG defines a global geometry of admissibility:

- configurations are points in a relational configuration space,
- admissibility is defined by a constraint-defined subset,
- deviation is measured by a scalar residual field,
- compatibility is encoded through residual-based geometry.

This structure is complete at the global level. Further geometric structure arises only when local behavior of projection is examined.

The residual field also suggests—but does not yet derive—the possibility of induced spectral structures on configuration space, such as kernel-based operators or Laplace-type constructions. These arise naturally in geometric and manifold-learning contexts, but their derivation from closure geometry requires additional local structure and is not pursued here.

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4. Interpretation and Role within Closure-First Frameworks

Projection Residual Geometry (PRG) provides a geometric characterization of admissibility in closure-first relational frameworks. It does not introduce new physical structure; rather, it makes explicit the minimal geometry required to relate arbitrary configurations to those satisfying global constraints.

4.1 PRG within the Ontic Minimalist Framework

Within the *Principles of Ontic Minimalism (OM)* [8], admissibility is defined by:

- global consistency (AGC),
- decomposition invariance (NPRF),
- closure (BBP).

PRG provides the minimal geometric structure required to operationalize this definition:

- admissible configurations form a constraint-defined manifold $\mathcal{M}$,
- arbitrary configurations are related to $\mathcal{M}$ via projection,
- deviation from admissibility is quantified by the projection residual.

Thus, PRG is not an independent theory. It is the **geometric realization of admissibility** within the ontic minimalist framework.

4.2 Global vs Local Structure

It is important to distinguish between:

- **global admissibility structure**, and
- **local restoration geometry**.

PRG addresses the global level:

- it defines $\mathcal{M}$,
- it defines projection onto $\mathcal{M}$,
- it defines the residual field $R(C)$.

It does not specify how admissibility is locally restored or maintained.

Local geometric structure—such as:

- connection-like behavior,
- path dependence,
- curvature and holonomy—

arises only when projection is examined in a neighborhood of $\mathcal{M}$. That refinement is developed in Closure-Restoration Geometry (CRG).

Thus:

PRG → global admissibility geometry
CRG → local geometric refinement

4.3 Relation to Closure-Restoration Geometry

The relationship between PRG and CRG is hierarchical.

PRG defines the global projection structure:

$$P_{\mathcal{M}}(C), R(C) \tag{4.1}$$

CRG develops the local geometric structure of this projection by examining how $P_{\mathcal{M}}$ behaves near admissible configurations.

In the near-admissible regime, CRG shows that:

- projection can be approximated by a local restoration map,
- the residual corresponds to least correction cost,
- variation of the restoration rule induces connection and curvature.

Thus:

$$\text{PRG residual} \longleftrightarrow \text{CRG local restoration cost} \tag{4.2}$$

More precisely:

CRG's construction is the local, first-order geometric development of the global projection structure introduced in PRG; the global residual $R(C)$and the local residual $R_\kappa(C)$ are consistent instantiations of the same concept at global and local scales.

4.4 Minimality and Ontic Status

PRG operates under the principle of minimality.

Projection introduces only:

- a distance relation,
- a constraint-defined subset,
- and a minimal correction rule.

It does not introduce:

- dynamical laws,
- hidden variables,
- or additional ontological layers.

In this sense, PRG is consistent with Local Projection Sufficiency (LPS) and No Ontic Excess (NOE):

- the admissible configuration is defined globally,
- local representations do not introduce additional structure,
- deviation from admissibility is fully captured by the residual.

The residual therefore has no independent ontic status—it is a geometric measure of constraint violation, not a physical field.

4.5 Representation and Neutrality

PRG is explicitly representation-neutral.

The following are representational choices:

- the ambient configuration space $\mathcal{C}$,
- the Grassmannian representation $\mathrm{Gr}_\rho(k, n)$,
- the metric d_ρ,
- and the projection operator $P_\mathcal{M}$.

These structures encode admissibility but do not constitute physical ontology.

Different representations may:

- encode the same constraints differently,
- define different coordinate systems,
- or use different metrics.

Under Representational Neutrality (RN), these differences do not correspond to distinct physical realities.

4.6 Relation to Physical Interpretation

PRG does not by itself specify a physical theory.

It provides:

- a geometric structure for admissibility,
- a measure of deviation from admissibility,
- and a framework for comparing configurations.

Physical interpretation arises only when:

- additional structure is introduced (e.g., capacity tensors),
- or when PRG is combined with local geometric refinement (CRG),
- or when specific sectors are analyzed (e.g., quantum conjugate sectors).

Thus PRG is foundational but not sufficient for physical prediction.

4.7 Summary

Projection Residual Geometry establishes the global geometric structure of admissibility:

- admissibility is projection onto a constraint-defined manifold,
- deviation is measured by a scalar residual,
- configuration space admits a natural geometric structure,
- no dynamical assumptions are required.

PRG therefore provides the minimal geometric foundation upon which more detailed structures—such as local geometry (CRG) and physical consequences (e.g., quantum structure)—are built.

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5. Summary and Forward Directions

Projection Residual Geometry (PRG) provides a minimal geometric formulation of admissibility in closure-first relational frameworks.

Starting from global consistency (AGC), decomposition invariance (NPRF), and closure (BBP), admissibility is expressed as projection onto a constraint-defined manifold $\mathcal{M}$. The projection operator $P_{\mathcal{M}}$ relates arbitrary configurations to admissible ones, and the projection residual $R(C)$ provides a scalar measure of deviation from closure compatibility.

In the Grassmannian representation, this induces a global configuration-space geometry:

- configurations correspond to relational subspaces $V \in \mathrm{Gr}_\rho(k, n)$,

- admissibility defines a subset $\mathcal{M} \subset \mathrm{Gr}_\rho(k, n)$,
- deviation is measured by a capacity-weighted metric d_ρ,
- and compatibility is encoded through residual-based structure.

This formulation introduces no dynamical assumptions and no additional ontological commitments. It provides the minimal geometric structure required to relate arbitrary configurations to those satisfying global constraints.

5.1 Position within the Framework

The role of PRG within the broader program is now explicit:

- **OM** establishes the principle architecture of admissibility and minimal realization (AGC, NPRF, BBP, LPS, NOE, RN).
- **PRG** provides the global geometric characterization of admissibility and deviation via projection and residual.
- **CRG** develops the local geometric refinement of projection, introducing restoration maps, connection-like structure, and curvature/holonomy.
- Subsequent work shows how physical structure (e.g., quantum conjugacy and uncertainty) arises from these foundations.

Thus, PRG is the **bridge layer**: it translates principle-level constraints into a global geometry on configuration space without yet invoking local differential structure.

5.2 Global–Local Continuity

PRG and CRG are consistent across scales. PRG defines the global projection $P_{\mathcal{M}}$ and residual $R(C)$. In neighborhoods of $\mathcal{M}$, CRG provides a first-order local realization of this projection as a least-cost restoration map.

Accordingly,

$$\text{global residual } R(C) \longleftrightarrow \text{local restoration cost } R_\kappa(C), \tag{5.1}$$

and both quantify the same deviation from admissibility at different levels of description.

5.3 Residual as Admissibility Boundary

The residual field Rdefines the admissibility boundary of configuration space:

$$\mathcal{M} = \{C \mid R(C) = 0\}. \tag{5.2}$$

Configurations with $R(C) > 0$ lie outside the closure-compatible set and require correction to satisfy global constraints.

This geometric interpretation replaces dynamical notions of "correction" or "evolution toward admissibility" with a purely structural criterion:

admissibility is location in configuration space relative to $\mathcal{M}$.

5.4 Compatibility Structure

The residual field induces a natural compatibility structure between configurations. A representative form is the residual-based kernel

$$K(V_i, V_j) \sim \exp\left(-\eta\, R(V_i, V_j)^2\right), \tag{5.3}$$

which measures relational compatibility without invoking transport or dynamics.

This structure is global and algebraic; it encodes how configurations align relative to admissibility, not how they evolve.

5.5 Scope and Limitations

PRG is intentionally limited to the global level. It does not:

- define local transport rules,
- introduce connection or curvature,
- or provide a dynamical theory.

These arise only when projection is analyzed locally (CRG). Likewise, while the residual field suggests the possibility of induced spectral or operator structures on configuration space, a derivation of such structures from closure geometry requires additional local analysis and is not undertaken here.

5.6 Outlook

Several natural directions follow:

1. **Local geometric refinement** — fully develop the connection, curvature, and holonomy structure of projection in CRG, including explicit hinge-level constructions and Grassmannian bundle formulations.
2. **Physical sector analysis** — apply the projection framework to specific sectors (e.g., conjugate variables), where admissibility geometry yields concrete physical consequences such as quantum uncertainty relations.

3. **Operator and spectral structure** — investigate whether residual-based kernels and associated operators can be derived canonically from closure geometry, rather than introduced analogically.

4. **Representation studies** — explore alternative representations of configuration space consistent with RN, to confirm that the projection–residual structure is representation-independent.

5.7 Final Statement

Projection Residual Geometry shows that admissibility in closure-first frameworks is inherently geometric. It is defined by projection onto a constraint-compatible manifold, and deviation from admissibility is measured by a scalar residual.

In this sense:

admissibility is not a process but a geometric condition,
and projection residual geometry provides its minimal global realization.

This establishes PRG as the foundational geometric layer upon which local structure (CRG) and physical consequences are built.

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